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Shock waves in non-Newtonian bubbly liquids [☆]

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Abstract

Propagation of non-stationary shock waves in a bubbly liquid with a non-Newtonian carrier phase was investigated numerically. The research is carried out within the one-velocity two-temperature two-pressure model of multiphase media mechanics. An aqueous solution of polymer was used as a non-Newtonian carrier phase. It was concluded that the behavior of shock waves in Newtonian and non-Newtonian bubbly liquids with the same Newtonian viscosity may have fundamental differences. The influence of defining parameters of the two-phase mixture and wave on the structure of shock waves was examined. The numerical analysis of evolution and attenuation of pulse perturbations was carried out. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bubbly liquid; Non-Newtonian; Polymer solution; Shock wave; Viscosity; Relaxation time

1. Introduction

The wave dynamics of bubbly systems with viscous and non-viscous Newtonian carrier phase is developed well enough for the present time (Gubaidullin et al., 1978; Nakoryakov et al., 1990; Gubaidullin, 1991; Nigmatulin, 1991). However, wave flows of those systems based on non-Newtonian liquids with complicated rheology (polymeric solutions and melts, suspensions, high-paraffinaceous, waxy and resinous, tar-content oils, etc.) have not been studied adequately in spite of their wide incidence in practice. Some problems of the behavior of a bubble in viscoelastic relaxing polymeric liquid and propagation of acoustic waves in the same liquid with bubbles are

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researched in this domain. These problems were best described by Levitskiy and Shulman (1995). The problems of dynamics of non-stationary nonlinear waves are not being studied.

This paper investigates the features of propagation of non-stationary shock waves in a non-Newtonian liquid with gas bubbles. Our intent is to compare the wave behavior in bubbly mixtures with non-Newtonian and viscous Newtonian carrier phases and to reveal the influence of defining parameters of two-phase mixture (Newtonian viscosity of solution and density of liquid, void fraction and size of bubbles, kind of gas, relaxation time, etc.) on the wave evolution process.

2. Basic equations

In this section we are going to obtain a system of equations, describing the dynamic behavior of the two-phase mixture mentioned above, taking into account the following assumptions (Gubaidullin et al., 1978): the radius of bubbles is many times larger than molecular-kinetic dimensions and many times smaller than the distances at which averaged or macroscopic parameters of mixture or phases vary essentially; the mixture is monodisperse; interaction, collisions of bubbles to one another may be neglected; processes of bubble fragmentation, coagulation and formation of new bubbles are absent; the pressure of gas in bubbles is homogeneous, velocities of macroscopic motion of phases are equal, the temperature and the density of carrier liquid are constant, phase transitions are absent.

2.1. The equations of motion of phases and of heat transfer

The continuity equations for gas, liquid and unit bubble and momentum equation at the assumptions mentioned above for non-stationary one-dimensional flow along axis x are as follows:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 v}{\partial x} = 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\partial \rho_2 v}{\partial x} = 0, \quad \frac{d}{dt}(\rho_2^0 a^3) = 0, \quad (1)$$

$$\rho \frac{dv}{dt} + \frac{\partial p}{\partial x} = 0 \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right). \quad (2)$$

Here ρ, p, v are, respectively, the density, the pressure and the velocity of mixture, $\rho_i^0, \rho_i, \alpha_i, p_i$ are the true and reduced densities, the void fraction, and the pressure of the i th phase, the subscripts $i = 1$ and 2 refer to liquid and gas parameters, a is the bubble radius.

The heat influx equation for gas may be written as

$$\rho_2 \frac{du_2}{dt} - \frac{\alpha_2 p_2}{\rho_2^0} \frac{d\rho_2^0}{dt} = nq_2, \quad T_1 = \text{const.}, \quad (3)$$

where T_i and q_2 are the temperature and heat influx to a bubble from liquid and n is the number of bubbles per unit volume of mixture. Within the two-temperature scheme (Gubaidullin et al., 1978) q_2 is given by

$$q_2 = 4\pi a^3 \frac{\lambda_2 Nu_2}{2a} (T_1 - T_2),$$

where λ_2 is the gas thermal conductivity and Nu_2 is the Nusselt number, and it can be expressed (Nigmatulin, 1991) as

$$Nu_2 = \begin{cases} 10, & Pe_2 \leq 100, \\ \sqrt{Pe_2}, & Pe_2 > 100, \end{cases}$$

$$Pe_2 = 12(\gamma_2 - 1) \frac{T_1}{|T_1 - T_2|} \frac{a|w|}{v_2^{(T)}},$$

$$v_2^{(T)} = \frac{\lambda_2}{\rho_2^0 c_{p2}}, \quad \gamma_2 = c_{p2}/c_{v2}.$$

Here $v_2^{(T)}$ is the gas thermal diffusivity, w the radial velocity of bubble interface, c_{p2}, c_{v2} the specific heats of gas at constant pressure and volume and Pe_2 is the Peclet number.

The gas is assumed to be ideal and the liquid is incompressible

$$p_2 = \rho_2^0(\gamma_2 - 1)c_{v2}T_2, \quad u_2 = c_{v2}T_2, \quad \rho_1^0 = \text{const.}$$

There are, by definition

$$\begin{aligned} \rho_1 &= \alpha_1 \rho_1^0, \quad \rho_2 = \alpha_2 \rho_2^0, \quad \rho = \rho_1 + \rho_2, \quad \alpha_1 + \alpha_2 = 1, \\ \alpha_2 &= \frac{4}{3} \pi a^3 n, \quad p = \alpha_1 p_1 + \alpha_2 (p_2 - 2\Sigma/a), \end{aligned}$$

where Σ is the surface tension of liquid.

2.2. Conditions of simultaneous strain of phases

As a condition of simultaneous strain of phases, a Rayleigh–Lamb equation for radial oscillations of a bubble in liquid is commonly used. The one generalized to the case of non-Newtonian liquid was obtained by Levitskiy and Shulman (1995)). It has the following form in a spherical coordinate system (r, φ, ψ) :

$$\begin{aligned} \rho_1^0 \left(a \frac{dw}{dt} + \frac{3}{2} w^2 \right) + p_1 - p_2 + \frac{2\Sigma}{a} &= 2 \int_a^\infty \frac{\tau^{(rr)} - \tau^{(\varphi\varphi)}}{r} dr, \\ w &= \frac{da}{dt}, \end{aligned} \tag{4}$$

where $\tau^{(rr)} - \tau^{(\varphi\varphi)}$ is the normal stress difference. The stress tensor $\hat{\tau}$ is defined by the rheological state equation of Maxwell type with the upper convective derivative and the single relaxation time t_τ (Astarita and Marrucci, 1974; Levitskiy and Shulman, 1995)

$$\begin{aligned} \hat{\tau} &= \hat{\tau}_s + \hat{\tau}_p, \quad \hat{\tau}_s = 2\eta_s \hat{e}, \quad \eta_s + \eta_p = \eta_0, \\ \hat{\tau}_p + t_\tau \left[\frac{d\hat{\tau}_p}{dt} - \frac{1}{2} (\hat{\tau}_p \cdot \hat{e} + \hat{e} \cdot \hat{\tau}_p) \right] &= 2\eta_p \hat{e}, \end{aligned}$$

where $\hat{\varepsilon}$ is the tensor of strain rate; the point means the internal multiplication of tensors of the second rank (multiplication of their matrices), η_0, η_s, η_p are Newtonian viscosity of solution (or zero-shear viscosity), the viscosities of solvent and of polymeric network in solution, respectively; the subscripts p and s refer to polymer and solvent.

The equations for normal components $\tau^{(rr)}, \tau^{(\varphi\varphi)}$ are written in the form (Levitskiy and Shulman, 1995)

$$\tau_p^{(rr)} + t_\tau \left(\frac{d\tau_p^{(rr)}}{dt} + 2\tau_p^{(rr)} \frac{a^2 w}{r^3} \right) = -4\eta_p \frac{a^2 w}{r^3}, \quad (5)$$

$$\tau_p^{(\varphi\varphi)} + t_\tau \left(\frac{d\tau_p^{(\varphi\varphi)}}{dt} - \tau_p^{(\varphi\varphi)} \frac{a^2 w}{r^3} \right) = 2\eta_p \frac{a^2 w}{r^3},$$

$$\tau_s^{(rr)} = -4\eta_s \frac{a^2 w}{r^3}, \quad \tau_s^{(\varphi\varphi)} = 2\eta_s \frac{a^2 w}{r^3}. \quad (6)$$

The expressions for $\tau^{(rr)}, \tau^{(\varphi\varphi)}$ can be found from (5) and substituted into (4). They give

$$\rho_1^0 \left(a \frac{dw}{dt} + \frac{3}{2} w^2 \right) + \frac{2\Sigma}{a} = p_2 - p_1 + S, \quad (7)$$

$$S = S_p + S_s,$$

$$S_p(t) = \frac{-4\eta_p}{t_\tau a^2(t)} \int_0^t e^{-(\xi-t)/t_\tau} a(\xi) w(\xi) d\xi, \quad (8)$$

$$S_s = -4\eta_s \frac{w}{a}. \quad (9)$$

2.3. The constitutive equation of liquid for radial flow around a bubble

The integral S_p can be transformed to the differential equation

$$\frac{dS_p}{dt} + \left(\frac{1}{t_\tau} + \frac{2w}{a} \right) S_p = -4 \frac{\eta_p}{t_\tau} \frac{w}{a}. \quad (10)$$

One can see from comparison of (10) and (9) with (5) and (6) that when $r = a$, $S_p = \tau_p^{(rr)}$, $S_s = \tau_s^{(rr)}$, i.e. the parameters S_p and S_s characterize the stresses on a bubble interface in polymer network and solvent, correspondingly (Levitskiy and Shulman, 1995).

Let us analyze, first, the limiting cases of Eq. (10). It will be assumed that t_* is the characteristic time of bubble oscillations, which determines the time scale of the process, S_{*p} is the scale of stress oscillations in polymer. Time t_* can be estimated, for instance, as a bubble collapse time in inertial Rayleigh regime $t_* \sim t_0 = a_0 \sqrt{\rho_{10}^0 / p_0}$. This estimation is valid when the dissipation is less than the inertia of radial movement of liquid around a bubble, i.e. oscillations of a bubble are possible. The estimates of terms of Eq. (10) are obtained:

$$\begin{aligned} \left| \frac{dS_p}{dt} \right| &\sim \frac{S_{*p}}{t_*}, & \left| \frac{1}{t_\tau} S_p \right| &\sim \frac{S_{*p}}{t_\tau}, \\ \left| 2 \frac{w}{a} S_p \right| &\sim \frac{S_{*p}}{t_*} & \left(\text{since } a \sim a_0, \text{ and } w = \frac{da}{dt} \sim a_0/t_* \right), \\ \left| -4 \frac{\eta_p}{t_\tau} \frac{w}{a} \right| &\sim \frac{\eta_p}{t_\tau t_*}. \end{aligned} \tag{11}$$

Small relaxation times. When $t_\tau \ll t_*$ the first and third terms in the left-hand side of Eq. (10), which contain time derivatives, may be neglected in comparison with S_p/t_τ . On both the right-hand side and the left-hand side of Eq. (10) there remains one term. These terms must be of the same order, i.e. $S_{*p} \sim \frac{\eta_p}{t_*}$. The following is the approximate expression for stress in polymer

$$S_p \approx -4\eta_p \frac{w}{a}, \quad t_\tau \ll t_*, \tag{12}$$

where Eqs. (9) and (12) allow one to write the expression for stress in solution as

$$S = S_p + S_s \approx -4\eta_0 \frac{w}{a}, \quad t_\tau \ll t_*. \tag{13}$$

Hence (12) and (13) have the same form as for a Newtonian liquid with the viscosity being equal to Newtonian viscosity of a polymer solution.

Large relaxation times. In the opposite case when $t_\tau \gg t_*$, it follows from (11) that one may ignore the second term S_p/t_τ in the left-hand side of (10) in comparison with the first and third terms, which involve derivatives. Thus, the following equation is obtained as

$$\frac{dS_p}{dt} + 2 \frac{w}{a} S_p = -4 \frac{\eta_p}{t_\tau} \frac{w}{a}, \quad t_\tau \gg t_*, \tag{14}$$

which taking into account the initial conditions ($t = 0 : S_p = 0, a = a_0,$) has the solution

$$S_p = 2 \frac{\eta_p}{t_\tau} \left[\left(\frac{a_0}{a} \right)^2 - 1 \right], \quad t_\tau \gg t_*, \tag{15}$$

It corresponds to the nonlinear-elastic returning force. When the bubble radius deviates to a small extent, (15) may be reduced to the following form:

$$\begin{aligned} S_p &\approx -4 \frac{\eta_p}{t_\tau} \frac{\Delta a}{a_0}, \quad t_\tau \gg t_*, \\ a &= a_0 + \Delta a, \quad |\Delta| a \ll a_0. \end{aligned} \tag{16}$$

Note that $\varepsilon^{(rr)} = -2\Delta a/a_0$ is the longitudinal strain in the liquid at bubble interface, so

$$S_p = \tau_p^{(rr)} \Big|_{r=a} \approx 2 \frac{\eta_p}{t_\tau} \varepsilon^{(rr)} \Big|_{r=a}.$$

Expressions (15) and (16) show that in this limiting case, the polymeric network in solution behaves as on elastic medium, with the elastic modulus $G = \eta_p/t_\tau$ determining the scale of stress oscillations $S_{*p} \sim \eta_p/t_\tau$. The total stress in solution consists of the elastic component due to polymer and the viscous one due to solvent:

$$S = S_p + S_s \approx 2 \frac{\eta_p}{t_\tau} \left[\left(\frac{a_0}{a} \right)^2 - 1 \right] - 4\eta_s \frac{w}{a}, \quad t_\tau \gg t_*. \quad (17)$$

Effective gas elastic modulus. It is obvious that the elastic stress in polymer will influence the behavior of a bubble if S_p exceeds the increment in gas pressure Δp_2 in a bubble due to Δa . Let us analyze this case. The simple polytropic law is as follows:

$$p_2/p_{20} = (\rho_2^0/\rho_{20}^0)^\kappa,$$

where κ is the polytropic index. Let

$$p_2 = p_{20} + \Delta p_2, \quad a = a_0 + \Delta a, \quad \text{where } p_{20} = p_1, \quad |\Delta p_2| \ll p_{20}, \quad |\Delta a| \ll a_0.$$

Since $\rho_2^0/\rho_{20}^0 = a_0^3/a^3$, then

$$\begin{aligned} \Delta p_2 &\approx -3\kappa p_{20} \frac{\Delta a}{a_0} = -4G_g \frac{\Delta a}{a_0}, \\ G_g &= \frac{3}{4}\kappa p_{20}, \end{aligned} \quad (18)$$

where G_g is the effective elastic modulus of the gas in a bubble. The elastic stress in polymer must be taken into account if

$$|S_p| \geq |\Delta p_2|, \quad \text{that is } G \geq G_g.$$

Note that the equilibrium value of pressure p_e behind the front of the wave should be used instead of p_{20} in (18) if applied to a shock wave of “step” type.

On the base of the performed analysis it may be argued that a polymeric liquid with bubbles for two limiting cases mentioned above behaves as a Newtonian bubbly liquid with viscosities η_0 and η_s , correspondingly, i.e. a shock wave might have two limiting structures at varying relaxation time: a shock wave traveling as in a Newtonian bubbly liquid with the viscosity being equal to Newtonian viscosity of solution $\eta_0(t_\tau \ll t_*)$ and a shock wave traveling as in a Newtonian bubbly liquid with the viscosity being equal to that of solvent $\eta_s(t_\tau \gg t_*)$. Modeling of polymeric solution by a Newtonian liquid is possible in these cases if the elastic modulus of polymeric network is less than the effective elastic modulus of the gas in a bubble.

3. Numerical results

The system of Eqs. (1)–(3) and (7)–(10) described above is solved numerically. The structure of the system differs from that for Newtonian bubbly liquid in one additional differential equation (10) for the stress in polymer. The latter involves only time derivatives, so the procedure proposed by Gubaidullin et al. (1978) for numerical modeling of Newtonian bubbly liquids can be applied. Following this procedure the system after conversion to the Lagrange coordinates is transformed so that the resulting system consists of one second-order differential equation on a space variable only, whereas the others are the first-order equations with respect to time. The modified Euler–Cauchy method is applied to solve the equations on time and the sweep method to solve the

boundary value problem for the second-order equation at each step on time. A more detailed description of the method numerical modeling is represented in Appendix A.

3.1. Simulation of stepwise shock wave

This research examines special features of those processes and the influence of defining parameters of the medium and initial parameters of the wave. The wave of step type is initiated by a drastic increase in pressure at the boundary of the medium and then the evolution of compression wave is considered.

Varying relaxation time. One of the most interesting simulation results is shown in Fig. 1, where the influence of relaxation time t_τ on structure of shock waves is examined. The profiles of non-dimensional pressure at $x = 20$ cm for various values of relaxation time $t_\tau = 0.001, 0.1$ and 1 ms are presented. Parameters of the mixture and the wave are $\rho_{10}^0 = 998$ kg/m³, $\eta_0 = 1.48$ Pa s, $\eta_s = 1$ mPa s, $a_0 = 1$ mm, $\alpha_{20} = 2\%$, $T_0 = 293$ K, $p_0 = 0.1$ MPa, $p_e = 0.3$ MPa. One can see from Fig. 1 that the amplitudes of oscillations significantly increase when the relaxation

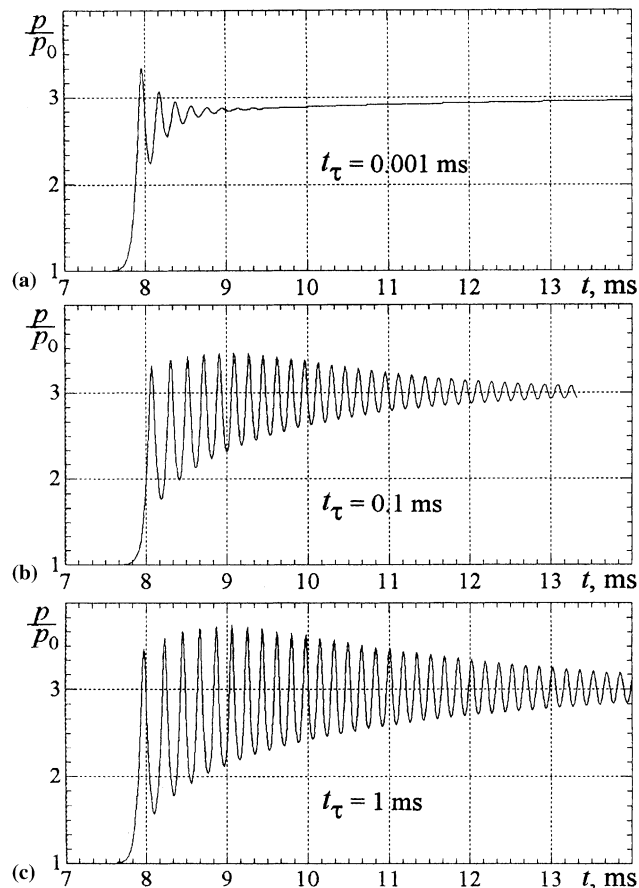


Fig. 1. Profiles of pressure in polymer liquid with air bubbles in shock wave for the various values of relaxation time t_τ .

time increases. A certain increase of oscillation period is also observed. The calculations show that the profiles of pressure at $t_\tau < 0.001$ ms practically coincide with each other and with the corresponding curve for the Newtonian carrier liquid with viscosity being equal to Newtonian viscosity of solution. In the other limiting case, $t_\tau \geq 1$ ms, the profiles of pressure coincide as well with the curve for the bubbly liquid with constant viscosity being equal to the viscosity of solvent. It fully corresponds to the previous analysis of limiting cases of Eq. (10). Characteristic Rayleigh time for the calculations discussed is $t_0 = a_e \sqrt{\rho_{10}^0/p_e} \approx 0.04$ ms. Characteristic time can be estimated also from the plots as reciprocal average cyclic frequency of oscillations $t_* \approx 1/\omega \approx 0.03$ ms, which is reasonably close to Rayleigh time.

Time t_* separates the limiting profiles in Fig. 1 on t_τ scale by nearly equal spaces in 1.5 decimal orders at each side of t_* (0.001, 0.03, 1 ms).

Newtonian and non-Newtonian carrying fluids. The behavior of shock waves in bubbly liquids with viscous Newtonian and non-Newtonian carrier phases is compared. Such computer analysis is carried out by the examples of glycerin and aqueous solution of polymer, the average molecular weight and concentration of which can be selected so that Newtonian viscosity of solution is equal to the viscosity of glycerin. The defining parameters of polymer solution are $\rho_{10}^0 = 998$ kg/m³, $\Sigma = 0.073$ kg/s², $\eta_0 = 1.48$ Pa s, $\eta_s = 1$ mPa s and $t_\tau = 1$ ms. We assume that the density and the surface tension of solution are close to those of the solvent (Ishiguro and Hartnett, 1992). The parameters of glycerin are $\rho_{10}^0 = 1260$ kg/m³, $\eta_N = 1.48$ Pa s. The evolution of “step” type shock wave of intensity $p_e = 0.15$ MPa in these liquids with air bubbles ($a_0 = 1$ mm, $\alpha_{20} = 2\%$, $T_0 = 293$ K, $p_0 = 0.1$ MPa) is presented in Fig. 2. It can be seen that the behavior of

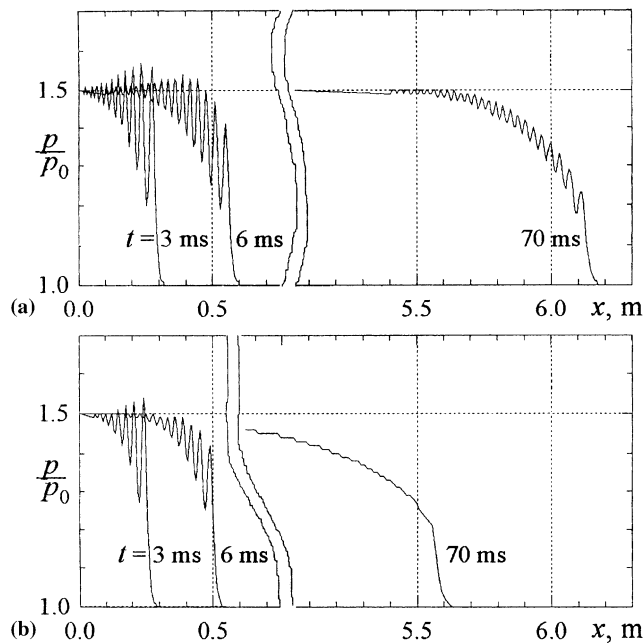


Fig. 2. Profiles of pressure in step-type shock wave in polymer solution and in glycerin with air bubbles. (a) Polymer. (b) Glycerine.

shock wave in these two cases is fundamentally different: the wave has a monotonous structure in the Newtonian bubbly liquid and oscillatory structure in the non-Newtonian one. As shown below, it occurs because in the non-Newtonian liquid, the effective viscosity becomes significantly lower than Newtonian viscosity of solution due to the oscillations of bubbles. It should be noted that the velocity of the shock wave in the first case is lower, because the density of glycerin is greater than that of the polymeric solution.

3.2. Simulation of nonlinear compression pulses

The character of damping of nonlinear compression pulses is studied. The results of computational investigation are illustrated by the example of compression waves with the initial semi-sinusoidal form. The picture of damping the same initial waves (intensity $p_e = 0.3$ MPa, initial duration of pulse is $10 \mu\text{s}$) in the non-Newtonian liquid (polymer solution, $t_r = 0.1$ ms, other parameters are as in Fig. 1) and in the Newtonian one (glycerin) with air bubbles is presented in Fig. 3. It may be noted that the character of damping varied, namely, in the non-Newtonian bubbly liquid it comes less intensively. It occurs due to the above-mentioned decrease of effective viscosity in the non-Newtonian liquid, which becomes much lower than its Newtonian viscosity, that is the viscosity of glycerin.

Let us inspect now how the value of relaxation time in polymer can influence the damping of pulses. We will consider the propagation of semi-sinusoidal pulses of different duration but of the fixed intensity 0.3 MPa in bubbly mixtures with various carrying phases: water, glycerin and aqueous polymer solutions ($\eta_0 = 1.48$ Pa s, $\eta_s = 1$ mPa s, $\rho_{10}^0 = 998$ kg/m³) with different relaxation times. Gas is air, $a_0 = 1$ mm, $\alpha_{20} = 0.5\%$, $p_0 = 0.1$ MPa, $T_0 = 293$ K. In Fig. 4 there are the integral damping curves of pulse disturbance of initial duration 1 ms in aqueous polymer solutions. One can see that with the relaxation time growing, the damping decreases. At the beginning of propagation, the pressure in the medium is higher than in initial pulse – this amplification is due to the effect of radial inertia. The analogous damping curve calculated for

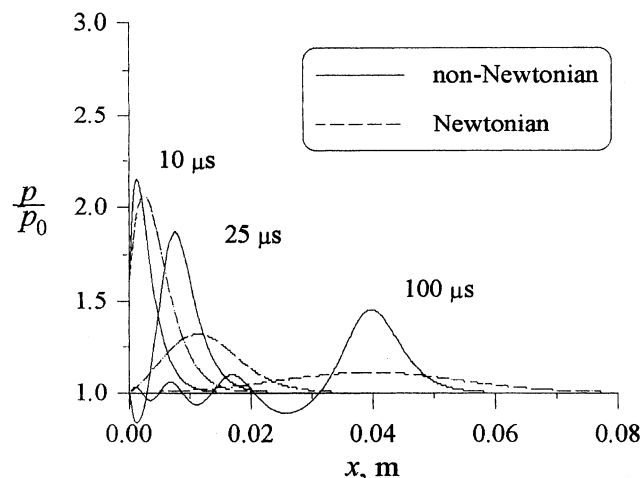


Fig. 3. Damping of compression pulse in non-Newtonian and Newtonian liquids with air bubbles.

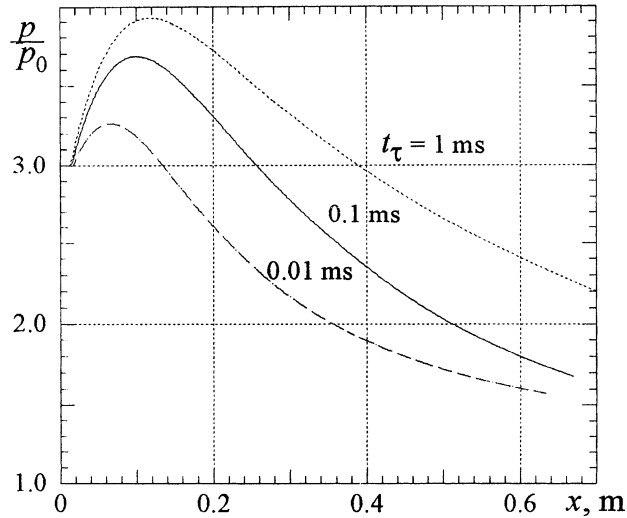


Fig. 4. The integral damping curves for semi-sinusoidal pulse in aqueous polymer solution with air bubbles for various relaxation times t_τ .

air–water mixture fully coincides with the dotted line ($t_\tau = 1$ ms) and that one for air–glycerin mixture does this with the dashed line ($t_\tau = 0.01$ ms). Thus, for pulse disturbances as for the stepwise shock waves, the conclusion remains true that in the limiting cases, when $t_\tau \gg t_*$ or $t_\tau \ll t_*$, the polymeric carrying phase behaves as a Newtonian liquid with the viscosities η_s and η_0 , respectively. This is true for the pulse disturbances independently on their initial duration.

Then we studied the influence of defining parameters of mixture on the pulse shock wave. Fig. 5 shows pressure profiles of the same damping pulse as in Fig. 4 but in different medium ($\eta_0 = 1.48$ Pa s, $\eta_s = 1$ mPa s, $t_\tau = 0.1$ ms) with air bubbles ($a_0 = 0.1$ mm, $\alpha_{20} = 1\%$, $p_0 = 0.1$

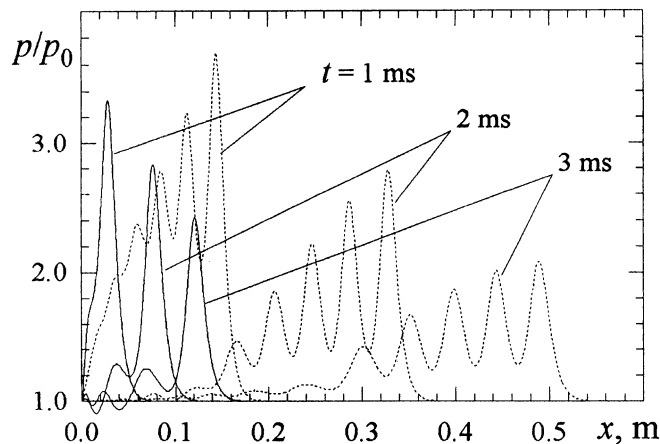


Fig. 5. Damping of semi-sinusoidal pulses in polymer solutions with air bubbles with different densities of carrying phase.

MPa) at two values of carrying phase density $\rho_{10}^0 = 998 \text{ kg/m}^3$ as in water (dotted lines) and $\rho_{10}^0 = 13546 \text{ kg/m}^3$ as in mercury (solid lines). It can be seen that the damping increases with the increase of density, which can be explained by the growth of thermal dissipation. In addition, the form of propagating pulses changes: they have a look of oscillating wave in one but a soliton look with an oscillating tail in the other mixture with a heavier liquid. Further calculations have shown that the influence of defining parameters of mixture and initial pulse (bubble radius, density and viscosity of carrying phase, duration of pulse) on the wave propagation in a polymeric bubbly liquid is qualitatively the same as in a Newtonian one. Non-Newtonian properties show themselves when the values of relaxation time are close to the characteristic time of bubble oscillations.

4. Analysis and discussion

4.1. Analysis of linear bubble oscillations

As it is known (Nigmatulin, 1991), the oscillatory profiles in the shock waves propagating through bubbly liquids appear due to bubble oscillations. To understand better the processes which might occur, a more thorough harmonic analysis of dynamic characteristics of a non-Newtonian liquid around an oscillating bubble will be carried out. Expressions for complex amplitudes can be obtained from Eqs. (9) and (10) in case of harmonic oscillations of bubbles with small real amplitude $\Delta a > 0$

$$a = a_0 + \Delta a \cdot \exp(i\omega t), \quad \Delta a \ll a_0. \tag{19}$$

The stresses S_p, S_s will be found as

$$S_k = S_k^* \exp(i\omega t), \quad k = p, s. \tag{20}$$

The velocity of the bubble interface w may be written as

$$W = i\omega \Delta a \exp(i\omega t),$$

i.e. the amplitude is

$$w^* = i\omega \Delta a, \tag{21}$$

then

$$\frac{w}{a} = \frac{i\omega \Delta a \cdot \exp(i\omega t)}{a_0 + \Delta a \cdot \exp(i\omega t)} \approx i\omega \frac{\Delta a}{a_0} \exp(i\omega t),$$

i.e.

$$\left(\frac{w}{a}\right)^* \approx i\omega \frac{\Delta a}{a_0}. \tag{22}$$

By substituting (19)–(22) into (10) and (9), one can obtain the expressions for the amplitude of stress in polymer S_p^* and in solvent S_s^*

$$S_p^* = -4\eta_p \frac{\Delta a}{a_0} \frac{i\omega}{1 + i\omega t_\tau}, \quad (23)$$

$$S_s^* = -4\eta_s i\omega \frac{\Delta a}{a_0}. \quad (24)$$

The last expression is a purely imaginary number that reflects the viscous response of the Newtonian solvent. The amplitude of the total stress is

$$S^* = S_p^* + S_s^* = -4i\omega \frac{\Delta a}{a_0} \left(\frac{\eta_p}{1 + i\omega t_\tau} + \eta_s \right). \quad (25)$$

The real and imaginary parts of $S_p^* = S_p' + iS_p''$ have the following form:

$$S_p' = -4 \frac{\eta_p}{t_\tau} \frac{\Delta a}{a_0} \frac{(t_\tau \omega)^2}{1 + (t_\tau \omega)^2}, \quad (26)$$

$$S_p'' = -4\eta_p \frac{\omega \Delta a}{a_0} \frac{1}{1 + (t_\tau \omega)^2}, \quad (27)$$

which describe the elastic and viscous responses of the liquid to the strain, respectively. Their ratio is

$$\frac{S_p'}{S_p''} = t_\tau \omega \sim t_\tau / t_*, \quad (28)$$

i.e. when $t_\tau \ll t_*$ ($t_\tau \omega \ll 1$) the imaginary part predominates

$$\left| S_p^* \right| \approx S_p'' \approx -4\eta_p \frac{\omega \Delta a}{a_0}, \quad (29)$$

and when $t_\tau \gg t_*$ ($t_\tau \omega \gg 1$), the real part predominates

$$\left| S_p^* \right| \approx S_p' \approx -4 \frac{\eta_p}{t_\tau} \frac{\Delta a}{a_0}. \quad (30)$$

Note that (29) and (30) are agreed with viscous and elastic limits (12) and (16).

4.2. Complex viscosities and moduli

The coefficient of the complex dynamic viscosity at small oscillations of a bubble in non-Newtonian liquid can be introduced in the same manner as for small shear oscillations (Ferry, 1980). In the Rayleigh–Lamb equation, the viscosity of a Newtonian liquid η_N is provided by the term

$$S_N = -4\eta_N w/a.$$

And the complex amplitude is

$$S_N^* = -4\eta_N i\omega \frac{\Delta a}{a_0},$$

thus

$$\eta_N = \frac{S_N^*}{-4i\omega\Delta a/a_0}.$$

We may introduce the complex dynamic viscosity in a way similar to that in the last expression

$$\eta^* = \eta' - i\eta'' = \frac{S^*}{-4i\omega\Delta a/a_0}. \tag{31}$$

Here η' is the real component of the complex η viscosity corresponding to the viscous response and η'' is the imaginary component corresponding to the elastic response.

Note that the strain rate of an incompressible liquid for the radial flow around a bubble is

$$e_{rr} = \frac{\partial v_r}{\partial r} = -\frac{2v_r}{r},$$

and its complex amplitude at bubble interface due to (22) is

$$e_{rr}^*|_{r=a} = -2i\omega \frac{\Delta a}{a_0}.$$

Thus, since $S = \tau^{(rr)}t|_{r=a}$,

$$\eta^* = \frac{\tau^{(rr)*}}{2e_{rr}^*}|_{r=a},$$

i.e. such introduction of the complex viscosity is natural.

For a polymeric liquid with single relaxation time from (24)–(27), we shall have

$$\eta^* = \frac{\eta_p}{1 + it_\tau\omega} + \eta_s, \tag{32}$$

$$\eta' = \frac{\eta_p}{1 + (t_\tau\omega)^2} + \eta_s, \quad \eta'' = \frac{\eta_p t_\tau\omega}{1 + (t_\tau\omega)^2}. \tag{33}$$

The expressions (33) agree with the expressions for components of the complex dynamic viscosity at periodic shear in Maxwell liquid with single relaxation time, except the additive to the real component corresponding to the solvent viscosity.

The plot of the dynamic viscosity components η' and η'' normalized to η_0 against $t_\tau\omega$ at $\eta_s/\eta_0 = 0.001/1.48$ is presented in Fig. 6 on a logarithmic scale. It is seen that the *real component* of viscosity η' monotonously decreases from Newtonian viscosity of solution η_0 to the viscosity of solvent η_s . Curve η' approaches to the horizontal asymptote $\eta = \eta_s$ proportionally by $1/(t_\tau\omega)^2$ while increasing $t_\tau\omega$.

The *imaginary component* of the complex dynamic viscosity η'' increases proportionally to $t_\tau\omega$ from zero to maximum $\eta'' = \eta_p/2$ at $t_\tau\omega = 1$, then decreases approaching zero as $1/(t_\tau\omega)$. It does not influence the structure of the wave, since the corresponding dynamic elastic modulus $G' = \omega\eta''$ ($G^* \equiv G' + iG'' = i\omega\eta^*$) is significantly lower than the reduced elastic modulus of gas in bubbles G_g (18), the dependence of which on $t_\tau\omega$ is shown in Fig. 6 via reduced imaginary viscosity of gas

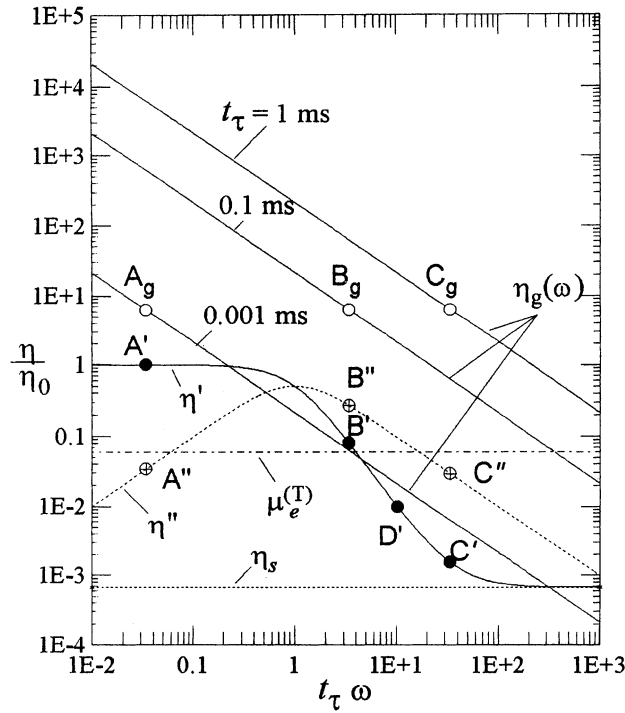


Fig. 6. The components of complex dynamic viscosity at bubble oscillations in Maxwell liquid vs. frequency ω multiplied by relaxation time t_τ .

$$\eta''_g(t_\tau \omega) = \frac{G_g}{\omega} = \frac{3}{4} \gamma_2 p_c t_\tau / (t_\tau \omega)$$

at various values $t_\tau = 0.001, 0.1$ and 1 ms.

From considered figure and formula (32), one can see that frequency dependencies of dynamic modules (viscosities) on $t_\tau \omega$ at the radial oscillation of bubbles within the Maxwell model have the same form as at periodic shear. Thus, the data about the spectra of dynamic modules or viscosities obtained by routine methods at periodical shear may be used for approximate analysis and estimations of parameters of bubble oscillations in real polymeric solutions.

4.3. Discussion of simulation results

The value of the viscosity realizing at bubble oscillations in polymeric solution in the shock wave shown in Fig. 2(a) is marked on the curve η' by point D' . This value is an order of two less than Newtonian viscosity of solution. It causes the appearance of oscillations in the shock wave traveling in the bubbly liquid with non-Newtonian carrier phase, unlike a Newtonian liquid with viscosity η_0 (Fig. 2(b)).

The plot of viscosity η' allows to interpret qualitatively the results of calculations illustrated in Fig. 1. The points A', B', C' on curve η' correspond to relaxation times and frequencies of bubble oscillations for the profiles a, b, c (Fig. 1). It should be noted that for small relaxation times

$t_r \omega \rightarrow 0$ (point A'), the viscosity realized is η_0 and the waveform coincides with the waveform in the Newtonian liquid with viscosity η_0 . For high relaxation times $t_r \omega \rightarrow \infty$ (point C'), the viscosity of the non-Newtonian liquid is close to the viscosity of solvent η_s and the waveform as in the Newtonian liquid with viscosity η_s is realized. The point B' corresponds to curve b (Fig. 1) and effective viscosity η' lies between η_0 and η_s . Since this case is rheologically nonlinear, the profile of the wave in mixture with Newtonian carrier phase with the same viscosity η' does not fully coincide with curve b . The simulations show that even if in some moment of evolution, the configurations of waves in non-Newtonian and Newtonian bubbly liquids with “effective” viscosity coincide, in the other moment of time they may be different.

The values of imaginary component of the viscosity corresponding to the wave profiles a, b, c (Fig. 1) are marked on curve η'' (Fig. 6) by points A'', B'', C'' . The points A_g, B_g, C_g show corresponding values of η''_g and as one can see from Fig. 6 they lie significantly higher than points A'', B'', C'' . They have approximately equal ordinates, that is the consequence of weak dependence of frequency of bubble oscillations in a shock wave on time of stress relaxation.

Effective thermal viscosity. It is known (Gubaidullin et al., 1978) that for liquids of low viscosity with fairly large bubbles, the main dissipative mechanism at bubble oscillations, which determines the evolution of the shock wave, is heat transfer between gas and liquid. To compare the contribution of this mechanism with dissipation due to viscosity of liquid one can use the effective thermal viscosity (Nigmatulin, 1991). For the results in Figs. 1 and 2, the effective thermal viscosity estimated is approximately $\mu_e^{(T)} \approx 0.06 \eta_0$. This was calculated with the formulas:

$$\mu_e^{(T)} = \mu_0^{(T)} \frac{a_e}{a_0} \sqrt{\frac{\omega_e v_{2e}^{(T)}}{\omega_0 v_{20}^{(T)}}},$$

$$\mu_0^{(T)} = \frac{3(\gamma_2 - 1)}{4\sqrt{2}} \rho_1^0 a_0 \sqrt{v_{20}^{(T)} \omega_0}, \quad \omega_0 = \frac{1}{a_0} \sqrt{\frac{3\gamma_2 p_0}{\rho_1^0}},$$

where a_e and $v_{2e}^{(T)}$ were used as adiabatic estimations of values behind the wave front

$$a_e = a_0 / (p_e / p_0)^{1/(3\gamma)},$$

$$v_{2e}^{(T)} = v_{20}^{(T)} / (p_e / p_0)^{1/\gamma},$$

and $\omega_e \approx 3.2 \times 10^4$ 1/s taken from Fig. 1. From Fig. 6, one can see that when the relaxation time is great, the evolution of wave is determined by thermal dissipation because $\eta'(C') \ll \mu_e^{(T)}$. In the other cases, the role of viscosity is comparable to the effect of inter-phase heat transfer (point A' and B'). In general, this relation varies. It depends on the time of stress relaxation and Newtonian viscosity of solution. This is confirmed by computer simulations made for mixtures with a different kind of gas in bubbles (air, carbon dioxide, helium, etc.).

5. Conclusion

The propagation of non-stationary shock waves in a bubbly liquid with a non-Newtonian carrier phase was investigated. A mathematical model describing the dynamic behavior of a

bubbly mixture with the non-Newtonian carrier phase was developed. The main conclusions are as follows:

1. The behavior of the shock wave in bubbly liquids with Newtonian and non-Newtonian carrier phase is fundamentally different: the wave may have a monotonous structure in a Newtonian bubbly liquid but an oscillatory structure in a non-Newtonian one with the same Newtonian viscosity. It occurs because the rheological losses may become much lower than the Newtonian viscosity due to the oscillations of bubbles.
2. Modeling of a polymeric bubbly liquid by a Newtonian one with “effective” viscosity so that the behavior of shock waves in these bubbly liquids is absolutely the same, is impossible in general case. But the polymeric bubbly solution can behave as a Newtonian bubbly liquid in two limiting cases:
 - (a) If the time of stress relaxation is much less than the time scale of bubble oscillations, then the behavior is like that of the Newtonian liquid with the viscosity being equal to Newtonian viscosity of solution.
 - (b) In the opposite limit, i.e. when the relaxation time is much greater than the time scale, the behavior is like that of the Newtonian liquid with the viscosity being equal to that of the solvent. This limit is purely realized only if the elastic response of polymer network is negligibly small as compared with that of the gas in bubbles; otherwise the polymer elasticity can change the behavior.
3. The attenuation of shock pulse in a non-Newtonian bubbly liquid is less than in a Newtonian one with viscosity being equal to Newtonian viscosity of solution.
4. The relation between thermal dissipation and dissipation due to viscosity of the carrier liquid is a variable value and it is defined by the time of stress relaxation and Newtonian viscosity of solution.
5. The frequency dependencies of dynamic modules (or complex viscosities) when radial oscillation of bubbles in the non-Newtonian liquid takes place, have the same form as in the case with periodical shear.

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Appendix A. Method of numerical modeling

The problem of a one-dimensional non-stationary flow of a single-velocity medium may be reasonably solved in Lagrangian coordinates (r, t) (Gubaidullin et al., 1978; Nigmatulin, 1991), where r is the distance from a particle to the origin at initial time $t = 0$. The parameter values at $t = 0$ are denoted by a subscript 0. In particular, $\rho_0 = \rho_0(r)$ is the mixture density at $t = 0$. The current position of the medium particle is characterized by the Eulerian coordinate $x(r, t)$, so that

$$\frac{\partial x}{\partial r} = \frac{\rho_0}{\rho}, \quad \frac{\partial x}{\partial t} = v. \quad (\text{A.1})$$

As a result of the assumed simplifications, the system of Eqs. (1)–(3), (7)–(10) of motion for a monodisperse mixture of an incompressible liquid with gas bubbles in the absence of phase transitions in the framework of a single-velocity two-temperature approximation assumes the following form in Lagrangian variables:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\rho_1 \rho}{\rho_0} \frac{\partial v}{\partial r} &= 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\rho_2 \rho}{\rho_0} \frac{\partial v}{\partial r} = 0 \quad (\rho = \rho_1 + \rho_2), \\ w &= \frac{\partial a}{\partial t}, \quad \frac{\partial}{\partial t} (\rho_2^0 a^3) = 0, \quad \alpha_2 = \frac{\rho_2}{\rho_2^0} = 1 - \frac{\rho_1}{\rho_1^0}, \quad \rho_1^0 = \text{const.}, \quad \rho_1 = \alpha_1 \rho_1^0, \\ a \frac{\partial w}{\partial t} &= \frac{p_2 - p_1 - 2\Sigma/a + S}{\rho_1^0} - \frac{3}{2} w^2, \\ S &= S_p + S_s, \quad S_s = -4\eta_s \frac{w}{a}, \quad \frac{\partial S_p}{\partial t} + \left(\frac{1}{t_\tau} + \frac{2w}{a} \right) S_p = -4 \frac{\eta_p}{t_\tau} \frac{w}{a}, \\ \frac{\partial v}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial r} &= 0, \quad p = p_1 \alpha_1 + \alpha_2 (p_2 - 2\Sigma/a), \\ \frac{\partial T_2}{\partial t} &= \frac{3(\gamma_2 - 1)}{2a_0^3 c_{v2} \rho_2^0} a \lambda_2 Nu_2 (T_1 - T_2) - 3(\gamma_2 - 1) \frac{w}{a}, \quad T_1 = T_0 = \text{const.}, \\ p_2 &= \rho_2^0 R_2 T_2. \end{aligned} \tag{A.2}$$

The generalized Rayleigh–Lamb equation (the fifth differential equation) may be transformed identically to a form in which, instead of average pressure p_1 , the reduced pressure p appears

$$a \frac{\partial w}{\partial t} = \frac{p_2 - p - 2\Sigma/a}{\rho_1} - \frac{3}{2} w^2 + \frac{S}{\rho_1^0}. \tag{A.3}$$

With the prescribed physical properties of liquid and those of gas (adiabatic index γ_2 , gas constant R_2 , coefficient of heat conductivity λ_2), and also with the given parameter of interphase heat exchange Nu_2 , and in the presence of initial and boundary conditions, the represented system of equations is closed.

In connection with the condition of incompressibility, we shall perform some identical transformations. The first two mass-conservation Eq. (A.2) of the system may be represented as

$$\frac{\partial \alpha_1}{\partial t} + \frac{\alpha_1 \rho}{\rho_0} \frac{\partial v}{\partial r} = 0, \quad \frac{\partial \alpha_2}{\partial t} + \frac{\alpha_2}{\rho_2^0} \frac{\partial \rho_2^0}{\partial t} + \frac{\alpha_2 \rho}{\rho_0} \frac{\partial v}{\partial r} = 0,$$

whence, upon summing these two equations, we obtain

$$\alpha_2 \frac{\partial \rho_2^0}{\partial t} + \frac{\rho_2^0 \rho}{\rho_0} \frac{\partial v}{\partial r} = 0.$$

Using the mass-conservation equation for the bubble (the fourth Eq. (A.2) in the form

$$a^3 \frac{\partial \rho_2^0}{\partial t} + 3a^2 \rho_2^0 w = 0,$$

we obtain that the longitudinal deformation of mixture with an incompressible continuous phase takes place only due to the bubble radial deformation

$$\frac{\partial v}{\partial r} = \frac{3\rho_0}{\rho} \frac{\alpha_2 w}{a}. \quad (\text{A.4})$$

In this case, equations governing variation of both mixture density and bubble volumetric concentration may be written as

$$\frac{\partial \rho}{\partial t} = -\frac{\rho^2}{\rho_0} \frac{\partial v}{\partial r} = -3\rho \frac{\alpha_2 w}{a}, \quad \frac{\partial \alpha_2}{\partial t} = \frac{\alpha_1 \rho}{\rho_0} \frac{\partial v}{\partial r} = \frac{3\alpha_1 \alpha_2 w}{a}. \quad (\text{A.5})$$

The relation between the Lagrangian and Eulerian variables is given as

$$\frac{1}{\rho} \frac{\partial v}{\partial x} = \frac{1}{\rho_0} \frac{\partial v}{\partial r}, \quad \left. \frac{d_2 \rho_2^0}{dt} = \frac{\partial \rho_2^0}{\partial t} \right|_r. \quad (\text{A.6})$$

Let us differentiate both the momentum equation with respect to r (recall that, $\rho_0 = \rho_0(r)$), and the continuity Eq. (A.4) with respect to t

$$\begin{aligned} \frac{\partial^2 v}{\partial r \partial t} &= -\frac{1}{\rho_0} \frac{\partial^2 p}{\partial r^2} + \frac{1}{\rho_0^2} \frac{d\rho_0}{dr} \frac{\partial p}{\partial r}, \\ \frac{\partial^2 v}{\partial t \partial r} &= 3\rho_0 \left(\frac{w}{\rho a} \frac{\partial \alpha_2}{\partial t} + \frac{\alpha_2}{\rho a} \frac{\partial w}{\partial t} - \frac{\alpha_2 w}{\rho^2 a} \frac{\partial \rho}{\partial t} - \frac{\alpha_2 w}{\rho a^2} \frac{\partial a}{\partial t} \right). \end{aligned} \quad (\text{A.7})$$

These two expressions in the domain of continuous motion may be set equal to each other. If, in this case, the time derivatives in the last equation are replaced by their values in accordance with both (A.5) and equations of radial motion, we obtain

$$\begin{aligned} \frac{\partial^2 p}{\partial r^2} - K \frac{\partial p}{\partial r} - Lp &= M, \quad K(r) = \frac{1}{\rho_0} \frac{d\rho_0}{dr}, \quad L = \frac{3\rho_0^2}{\rho_1^0} \frac{\alpha_2}{\alpha_1 a^2 \rho_1^0}, \\ M &= \frac{3\rho_0^2 \alpha_2}{\rho a^2} \left\{ \frac{p_2 - 2\Sigma/a}{\alpha_1 \rho_1^0} + \frac{S}{\rho_1^0} + 3 \frac{w^2}{2} + 2w^2 \right\}. \end{aligned} \quad (\text{A.8})$$

Thus, the condition of incompressibility of continuous phase, as well as in hydrodynamics of a single-phase incompressible liquid, led to a second-order differential equation which contains derivatives with respect only to the spatial coordinate. This equation makes it possible to determine the pressure distribution at any point in time provided the remaining parameters (S, α_2, a, w , and p_2) are prescribed at the same point in time.

It should be kept in mind that pressure p distribution and velocity v distribution at each point in time, including $t = 0$, are not independent variables because of the incompressibility of the continuous liquid. The p distribution is determined from the above-indicated boundary-value problem in terms of distributions of α_2, a, w, S and p_2 and also in terms of boundary conditions on the ends $r = (0, L)$, and the distribution of velocity (v) is found from Eq. (A.4), or from the momentum equation. The variation of α_2, a, w, S and p_2 in time is defined by differential equations (containing only time derivatives) following from Eqs. (A.2) and (A.5).

Let us introduce dimensionless variables denoted by bars

$$\bar{p}_i = \frac{p_i}{p_0}, \quad \bar{a} = \frac{a}{a_0}, \quad \bar{v} = \frac{v}{C_*}, \quad \bar{w} = \frac{w}{w_*}, \quad \bar{x} = \frac{x}{L_\alpha}, \quad \bar{T}_i = T_i/T_0, \quad \bar{\rho}_i^0 = \rho_i^0/\rho_{i0}^0, \quad (A.9)$$

$$\bar{C}_{10} = \frac{C_1}{C_\alpha}, \quad \left(C_* = \sqrt{\frac{p_0}{\rho_1^0}}, \quad C_\alpha = \frac{C_*}{\sqrt{\alpha_{10}\alpha_{20}}}, \quad L_\alpha = \frac{a_0}{\sqrt{\alpha_{10}\alpha_{20}}} \right),$$

$$w_* = a_0/t_*, \quad t_* = a_0/C_*, \quad \lambda_2^* = \frac{\lambda_2}{\rho_{20}^0 c_{20} a_0 C_*}, \quad \eta_0^* = \frac{\eta_0}{a_0 \rho_{10}^0 C_*}, \quad (A.10)$$

$$\Sigma_0^* = \frac{\Sigma}{a_0 p_0}, \quad S_* = p_0.$$

To facilitate both the analysis and solution of the system of equations being discussed, we shall use dimensionless variables (A.9) together with parameters (A.10) denoted, as before, by a bar ($\bar{p}, \bar{\rho}, \bar{\rho}_2^0, \bar{a}$); the latter are determined as a ratio of parameters corresponding to their characteristic values (scales) designated by a subscript 0. Some fixed values ($P_0, \rho_1^0, \rho_{20}^0, a_0$), being characteristics for distributions of p, ρ, ρ_2^0 , and a at the initial time $t = 0$, are chosen as scale factors. The latter, for convenience, are selected so that they satisfy the conditions of equilibrium

$$p_{20} = p_{10} + 2\Sigma/a_0, \quad \rho_{20}^0 = p_{20}/(R_2 T_0). \quad (A.11)$$

As before (see (A.9)), the equilibrium speed of sound $C_e = C_\alpha$ is chosen as the inherent velocity of the longitudinal motion, and the speed C_* is chosen as the inherent velocity of radial motion near bubbles. The linear scale L_α is adopted the same as in (A.9). Accordingly, the time scale factor t_0 is selected as

$$\bar{t} = \frac{t}{t_0}, \quad \bar{r} = \frac{r}{L_\alpha} \quad \left(t_0 = \frac{L_\alpha}{C_\alpha} = \frac{a_0}{C_*} \right).$$

As a consequence, the system of equations may be reduced to the following form (let us drop the sign “-” above dimensionless variables)

$$\frac{\partial^2 p}{\partial r^2} - K \frac{\partial p}{\partial r} - Lp = -M, \quad (A.12)$$

$$\frac{\partial \Phi_i}{\partial t} = \Pi_i \quad (i = 1, 2, 3),$$

$$\Phi_1 = w, \quad \Phi_2 = a, \quad \Phi_3 = T_2,$$

$$\Pi_1 = \frac{1}{a} \left[\frac{p_2 - p - 2\Sigma/a}{\alpha_1} - 1.5w^2 + S \right], \quad S = S_S + S_p, \quad S_S = -4\eta_s \frac{w}{a},$$

$$\frac{\partial S_p}{\partial t} + \left(\frac{1}{t_\tau} + \frac{2w}{a} \right) S_p = -4 \frac{\eta_p}{t_\tau} \frac{w}{a},$$

$$\Pi_2 = w, \quad \Pi_3 = -\frac{3(\gamma_2 - 1)}{a} w T_2 + \frac{3\lambda_2}{2\rho_{20}^0 c_{v2}} a N u_2 (1 - T_2),$$

$$K = \frac{1}{\rho_0} \frac{d\rho_0}{dr}, \quad L = \frac{3\rho_0^2}{\rho} \frac{\alpha_2}{\alpha_1 a^2},$$

$$M = \frac{3\rho_0^2 \alpha_2}{\rho a^2} \left\{ \frac{p_2 - 2\Sigma/a}{\alpha_1} + S + 3 \frac{w^2}{2} + 2w^2 \right\}.$$

Note that in the case of one-dimensional flow with plane waves, the velocity of the medium appears only in the second Eq. (A.4); therefore, to calculate it on each step of integration with respect to time is not necessary; and, in cases when the v distribution must be calculated, it is better to do it for each time interval of integration following the computation of p distribution based upon the momentum equation which in dimensionless variables is written in the form

$$\frac{\partial v}{\partial t} = -\frac{\alpha_{10}\alpha_{20}}{\rho_0} \frac{\partial p}{\partial r}.$$

The obtained system consists of six differential equations, each of which contains only a single derivative with respect to one of the coordinates (r or t). The first equation of the system is intended for determining the reduced pressure at an arbitrary time using the known fields of the rest of the parameters; the remaining equations describe the laws of variation of Lagrangian particles of the medium in time.

For numerical integration of the obtained system of equations, we shall divide the selected volume of the medium by points $r = r_i$ ($i = 1, 2, \dots, n$) into n material particles: the values of all sought functions we determine at points $r = r_i$ ($i = 1, 2, \dots, n$). Then, the last five differential equations in partial time-derivatives of variables a, w, S_p, T_2, v are transformed into $5n$ ordinary differential equations with respect to time, for whose numerical integration, the modified Euler–Cauchy method may be conveniently used. Corresponding difference equation can be written in the form

$$\begin{aligned} \left(\tilde{\Phi}_i\right)_m^n &= \left(\Phi_i\right)_m^n + \left(\Pi_i\right)_m^n \Delta t, \\ \left(\Phi_i\right)_m^{n+1} &= \left(\Phi_i\right)_m^n + \left[\left(\tilde{\Pi}_i\right)_m^n + \left(\Pi_i\right)_m^n\right] \Delta t / 2. \end{aligned}$$

Here n and m superscript and subscript correspond to the parameters in m – sites of difference network at time moment n ; $\tilde{\Pi}_i = \Pi_i(\tilde{\Phi}_j)$; Δt is step of integration on time.

To find the pressure p values at points $r = r_i$ at each fixed point in time, a linear (for p) boundary – value problem must be solved for the first differential equation (with respect to r) of the second order with boundary conditions

$$\begin{aligned} \frac{\partial^2 p}{\partial r^2} - K \frac{\partial p}{\partial r} - Lp &= -M, \quad 0 < r < L, \\ l_0 p(0, t) + l_1 \frac{\partial p}{\partial r}(0, t) &= \psi_0(t), \\ d_0 p(l, t) + d_1 \frac{\partial p}{\partial r}(l, t) &= \psi_l(t). \end{aligned}$$

To facilitate the solution of this problem the sweep method is advisable, in accordance to which the boundary-value problem is reduced to the solution of set of linear algebraic equations (the superscript n related to the network function will be dropped) by means of approximating the derivatives with respect to r by central differences

$$p_{m-1} - A_m p_m + B_m p_{m+1} = F_m, \quad m = 1, 2, \dots, s-1,$$

$$p_0 = \kappa_1 p_1 + \eta_1,$$

$$p_s = \kappa_2 p_{s-1} + \eta_2.$$

The sweep method consists of two steps – the direct and reverse ones. The direct step consists in calculating sweep coefficients δ_m and β_m :

$$\delta_{m+1} = \frac{B_m}{A_m - \delta_m}, \quad \beta_{m+1} = \frac{\beta_m - F_m}{A_m - \delta_m}, \quad m = 1, 2, \dots, s-1,$$

$$\delta_1 = \kappa_1, \quad \beta_1 = \eta_1.$$

The reverse step consists in solving the problem

$$p_s = \frac{\kappa_2 \beta_s + \eta_2}{1 - \kappa_2 \delta_s}, \quad p_m = \delta_{m+1} p_{m+1} + \beta_{m+1},$$

$$m = s-1, s-2, \dots, 0.$$

The advantage of the sweep method is the small number of algebraic operations and the low sensitivity to the error of calculation. It should be noted, for comparison, that the shorting method is inapplicable in this case because of its instability.

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